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Interfaces of Ground States in Ising Models with Periodic Coefficients

Luis A. Caffarelli[1] and Rafael de la Llave[1]

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We study the interfaces of ground states of ferromagnetic Ising models with external fields. We show that, if the coefficients of the interaction and the magnetic field are periodic, the magnetic field has zero flux over a period and is small enough, then for every plane, we can find a ground state whose interface lies at a bounded distance of the plane. This bound on the width of the interface can be chosen independent of the plane. We also study the average energy of the plane-like interfaces as a function of the direction. We show that there is a well defined thermodynamic limit for the energy of the interface and that it enjoys several convexity properties.

KEY WORDS: Interfaces; Ising models.

1. INTRODUCTION

The goal of this paper is to study the interfaces of Ising models in which the material has a periodic structure and is subject to a weak magnetic field, also spatially periodic with the same period, with zero mean flux and which is not too strong.

Roughly speaking, we will show (see Theorem 2.1 for a precise statement) that such models possess ground states whose interface is plane-like (i.e., contained between two parallel planes). The orientation of these interfaces is arbitrary and furthermore the width of the strip containing the interface can be chosen to be independent of the orientation. We will also show that there is a well defined limit of the average energy of the interface and that this limit satisfies convexity properties.

[1]University of Texas at Austin, Austin, TX 78712-1082, USA; e-mail: caffarel@math.utexas.edu; llave@math.utexas.edu

Results similar to those above have been proved for minimal surfaces in ref. 4. The results presented here are very similar to those above because the energy of an Ising model is closely related to the area of the interface. Indeed, the proofs presented here follow the same strategy as those in ref. 4.

Nevertheless, because of the discrete nature of the model, some of the technical arguments are easier. In particular, the density estimates needed in ref. 4 are trivial for the models in the present paper. In contrast, the arguments related to calculus are not present for the present models. Indeed, as we will see, several of the results that are established for continuous models in refs. 5 and 6 are false even for the standard Ising model. Notably, for the continuum models, all the minimizers have plane-like interfaces, while this is not the case for Ising models. The minimizers of the discrete case may not satisfy any maximum principle.

2. NOTATION AND STATEMENT OF RESULTS

2.1. Notation on Ising Models

We refer to refs. 10, 16, 17 for more information on statistical mechanics models. Nevertheless, in this paper, we will consider only ground states – zero temperature – and we will include most of the notation that we use.

The Ising models we will consider will be defined on a lattice $\mathbb{Z}^d$ which, for convenience in some geometric arguments, we will consider as contained in $\mathbb{R}^d$. We will consider the lattice endowed with the usual ℓ^1 distance, $|k| = \sum_{i=1}^{d} |k_i|$.

A configuration s is a mapping $s\colon \mathbb{Z}^d \to \{+1, -1\}$. We will denote by $\mathcal{C}$ the space of configurations.

Given a configuration s, we will denote by ∂s the interface of the configuration. That is

$$\partial s = \{i \in \mathbb{Z}^d \mid s_i = +1, \exists\, j \quad \text{s. t. } |i-j| = 1, s_j = -1\} \tag{1}$$

The behavior of an Ising model is described by a (formal) functional on configurations.

$$H(s) = \sum_{\substack{i,j \in \mathbb{Z}^d \\ |i-j| \leqslant R}} J_{ij}(s_i s_j - 1) + \sum_{i \in \mathbb{Z}^d} h_i s_i \tag{2}$$

In the classical Ising model, $J_{ij}=1$ but in this paper, we want to consider more general models, in particular, we do not want to keep translation invariance by all vectors, even if we will assume periodicity by some sub-lattice of vectors.

Given $\omega \in \mathbb{R}^d$, we denote $\Pi_\omega = \{x \in \mathbb{R}^d | \omega \cdot x = 0\}$. Clearly, $\Pi_\omega = \Pi_{\omega'}$ when ω is a multiple of ω'.

Remark 2.1. Some of the results that we will discuss go through for somewhat more general models in which the interaction may be three or more bodies or the lattice does not need to be an Euclidean lattice but rather in a richer geometric framework considered in ref. 3.

We will not discuss such generalizations here, nevertheless, we point out that these generalizations could be necessary to make contact with continuum models.

The number R is referred to as the range of the interaction. In the classical Ising models, the range is 1, which corresponds to only nearest neighbor interactions. In this paper, we will only consider finite range interactions, but the existence results go through for infinite range interactions by taking limits.

Given a set $\Gamma \subset \mathbb{Z}^d$ and a number R we denote Γ^R the set of points in Γ whose distance to $\mathbb{Z}^d$ is smaller or equal than R. When R is the range of the interaction, Γ^R is the collection of sites that can interact with the sites in Γ.

Given a finite set $\Gamma \subset \mathbb{Z}^d$, we define

$$H_\Gamma(s) = \sum_{\substack{i \in \Gamma,\, j \in \mathbb{Z}^d; i \in \mathbb{Z}^d\, j \in \Gamma \\ |i-j| \leqslant R}} J_{ij}(s_i s_j - 1) + \sum_{i \in \Gamma} h_i s_i. \tag{3}$$

A very important definition for us is

Definition 2.1. We say that a configuration s is a ground state when

$$H_\Gamma(u) \geqslant H_\Gamma(s)$$

for all u that agree with s in $(\mathbb{Z}^d - \Gamma)^R$.

Note that Definition 2.1 only uses finite sums, so that the formal character of the sums (2), does not matter. We also note that the notion of ground state – quite customary in Physics – is also very similar to the notion of class A minimizer in ref. 14.

We recall that there is an equivalent description of the energy of Ising models, which makes the connection with geometric questions clearer, namely, the description of a state in terms of contours.

A configuration can be described by specifying the set

$$\mathcal{S}(s) = \{ j \in \mathbb{Z}^d \mid s_j = +1 \} \tag{4}$$

As it is customary in statistical mechanics, the boundary of the set $\mathcal{S}$ can be described geometrically by placing a unit plaque perpendicularly across each bond joining $\mathcal{S}$ with its complement.

Notice that ∂s, the interface of the configuration s, is very similar to the boundary of the set $\mathcal{S}(s)$ and the functional is very similar to the area of the plaques.

Remark 2.2. In the language of contours, the theory of ground states is very similar to the theory of minimal surfaces as formulated in the language of *sets of finite perimeter*.

As an illustrative example, when $J_{ij} = 1$

$$\sum_{\substack{|i-j|=1 \\ i,j \in \mathbb{Z}^d}} J_{ij}(s_i s_j - 1),$$

is twice the area of the contour describing $\mathcal{S}$, and ground states of systems without magnetic field, correspond to surfaces whose area cannot be decreased by making local modifications. Hence, ground states can be considered as discrete analogues of minimal surfaces.

The terms involving h correspond to volume terms in the set. In the continuum case, the stationary points for the variational principle $\mathrm{Per}(\Omega) + \int_\Omega h$ have boundaries which are solution of the prescribed curvature equation. (if $x \in \partial\Omega$ the mean curvature of $\partial\Omega$ at x is precisely $h(x)$). See ref. 8.

There is a theory of minimal surfaces based on studying surfaces as boundaries of sets (an account of this theory can be found in ref. 8 and it was the basic language of ref. 4.) The analogue of the sets of finite perimeter in the geometric theory is the sets $\mathcal{S}(s)$ associated to the configurations.

As it turns out, the proofs of several of the results will be follow the strategy for the results in ref. 4, which were formulated in the language of sets of locally finite perimeter. Of course the details of the proofs will have to be different since many methods from calculus are not available. Indeed, in Section 5, there are examples that show that the straightforward analogues of some of the results in refs. 5, 6 are false for the models that we consider here.

Remark 2.3. We recall that there are two physical interpretations that are reasonable for these models. One is that the s_i are the states of spin of an atom at site i. The other – usually called lattice gases – is that the s_i describe whether a site is occupied or not. In the first interpretation, the average energy of the ground state has the interpretation of a magnetic energy near a wall. In the second, it is a surface tension. The physical interpretation in terms of lattice gases is very close to the mathematical framework in terms of sets of smallest perimeter.

2.2. The assumptions of this paper

We will consider systems of the form (2) such that they are

H1. Periodic of period N. That is:

$$J_{i+e,\, j+e} = J_{ij} \quad \forall\ e \in N\mathbb{Z}^d$$
$$h_{i+e} = h_{i+e} \quad \forall\ e \in N\mathbb{Z}^d$$

H2. – Weakly ferromagnetic. That is:

$$J_{ij} \leqslant 0.$$

– There is a $c < 0$ such that for each site i, there is one j such that

$$J_{ij} \leqslant c.$$

H3. The magnetic field h has zero flux

$$\sum_{i \in F} h_i = 0$$

where F is a fundamental domain for $\mathbb{Z}^d / N\mathbb{Z}^d$. That is, $F = \{0, 1, \dots, N-1\}^d \subset \mathbb{Z}^d$.

H4. $h^* \equiv \sup |h_i|$ sufficiently small, depending on the properties of the model considered above.

Remark 2.4. Sometimes, in statistical mechanics one uses a hypothesis significantly stronger that *H2*, namely

There is a $c<0$ such that for each site i, for all the j such that $|i-j|=1$,

$$J_{ij} \leqslant c$$

Which is clearly satisfied by the classical Ising model. This hypothesis does not lead to improvements in our results. Of course, this hypothesis is crucial in many of the studies in the literature which consider properties of the models that we will not examine.

The first main result of this paper is the following

Theorem 2.1. Given any Ising model satisfying *H1, H2, H3, H4* above there is M such that for every hyperplane $\Pi_\omega \subset \mathbb{R}^d$ of normal vector ω, we can find a ground state s_ω whose interface ∂s_ω is contained in a strip of width M around the plane Π_ω.

That is:

$$d(\partial s_\omega, \Pi_\omega) \leqslant M \tag{5}$$

We will show that the M can be chosen to depend on very few properties of the model, it only depends on the dimension, the range of the interaction, the periodicity, $\max|J_{i,j}$, $\min|J_{i,j}$, $\max|h_i$.

As we will see later, there are other properties which we will prove about the interface of the ground states which appear in the conclusions of Theorem 2.1 notably that the interfaces satisfy a so-called "Birkhoff property" (see Proposition 3.4) which plays an important role in Aubry-Mather theory. It was introduced in refs. 1, 11–13. Similar properties appear in the study for geodesics on surfaces in refs. 9, 14. For PDE's this property was considered in ref. 15. As it turns out, the Birkhoff property for some minimizers does not need even the full H2 and it suffices that the system is weakly ferromagnetic.

Remark 2.5. Note that Theorem 2.1 only claims that there exists ground states satisfying the conclusion. As we will see, even for the classical Ising model in $d=2$, when the interface is not oriented along the coordinate axis, it is possible to obtain ground states which do not satisfy (5).

We will refer to ground states satisfying (5) for some M as plane-like ground states. Note that in Theorem 2.1 we show that the M can be chosen uniformly for all orientations.

We will also prove another result giving the existence of an average interface energy for all the plane like minimizers.

Theorem 2.2. In the assumptions of Theorem 2.2.

Let Σ be a compact set of $\mathbb{R}^d$ with C^1 boundary. For $\lambda \in \mathbb{R}^+$, Denote by $\lambda\Sigma = \{x \in \mathbb{R}^d | (1/\lambda)x \in \Sigma\}$.

For $\omega \in \mathbb{R}^d$, $|\omega| = 1$,

Let s be a ground state whose interface lies at a bounded distance from the plane Π_ω.

Then, we have:

$$\lim_{\lambda \to \infty} H_{\lambda\Sigma}(s)/\lambda^{d-1} = |\Sigma \cap \Pi_\omega|_{d-1} \mathcal{A}(\omega) \tag{6}$$

where $| \quad |_{d-1}$ denotes the $d-1$ surface area.

Note that $\mathcal{A}(\omega)$ is independent of Σ and s. It is a property only of the model.

Moreover, the function $\mathcal{A}$, when extended to $\mathbb{R}^d$ as a positively homogeneous function of degree 1 (i.e., $\mathcal{A}(\lambda\omega) = \lambda\mathcal{A}(\omega)$ for $\lambda \in \mathbb{R}^+$) is convex.

The limit in (6), is reached very uniformly. If Σ has C^1, boundary, there exists a constant Ω_Σ depending on Σ – and of course, the properties of the model – but independent of ω, s such that

$$|H_{\lambda\Sigma}(s)/\lambda^{d-1} - |\Sigma \cap \Pi_\omega|_{d-1} \mathcal{A}(\omega)| \leqslant \Omega_\Sigma \lambda^{-1/2} \tag{7}$$

The exponent $-1/2$ in the remainder in (7) is not optimal. Also it seems that one can relax the regularity requirements on the surface Σ. The only thing required is that one can approximate it well by cubes.

The physical meaning of $\mathcal{A}(\omega)$ is the density of magnetic energy of the interface. In the lattice gas interpretation of the model, $\mathcal{A}(\omega)$ is a surface tension. The homogeneity is natural if we think of ω as being a "*surface element*". That is a vector oriented along the normal and with modulus the area.

We note that $\mathcal{A}(\omega)$ is also related to the average action in Aubry–Mather theory or to the stable-norm in the calculus of variations. Note, however that the discrete nature of the problem makes it impossible to use many of the arguments customary in these theories. Indeed, some of the results obtain in the continuous cases are false for the discrete cases considered here.

3. PROOF OF THEOREM 2.1

The strategy of proof will be very similar to that of ref. 4. We will establish the existence of some particular minimizers (the infimal minimizers) for rational ω. We will show that the interfaces of these infimal minimizers satisfies uniform bounds that we will be able to pass to the limit of irrational frequencies.

The first step will be to consider minimizers among configurations which are periodic and which satisfy some constraints. Among them, we will consider a particular one, which will enjoy special properties.

3.1. Notation

First we will develop some notation which will allow us to work comfortably with translations, periodicities, fundamental domains, multiplying fundamental domains, etc.

3.1.1. Translations

We introduce the translation operators $\mathcal{T}_k$, $k \in \mathbb{Z}^d$ acting on configurations

$$(\mathcal{T}_k s)_{i+k} = s_i \quad \forall i.$$

An important property of the models as in (2) satisfying periodicity is that formally for all configurations s,

$$H(\mathcal{T}_k s) = H(s) \quad \forall k \in N\mathbb{Z}^d$$

in the sense that all the terms that appear on one side appear on the other.

A precise form of the above is that, for every finite set Γ and for every configuration s we have

$$H_\Gamma(\mathcal{T}_k s) = H_{\Gamma+k}(s) \quad \forall k \in N\mathbb{Z}^d. \tag{8}$$

Equation (8) can be established readily noting that it is just a change in the dummy variables in the sum.

For sets Γ, we introduce the notation

$$\mathcal{T}_k \Gamma = \Gamma + k$$

Note that this consistent with the application of $\mathcal{T}_k$ to the characteristic function of Γ. With this notation, (8) can be written as

$$H_{\mathcal{T}_k\Gamma}(\mathcal{T}_k s) = H_\Gamma(s).$$

3.1.2. Symmetries

From now on and until further notice, we will consider $\omega \in (LN)^{-1}\mathbb{Z}^d)$ where N is the period of the model and $L \in \mathbb{N}$. The frequencies of $N^{-1}\mathbb{Z}^d$ are the frequencies that correspond to planes in the lattice given by fundamental domains of the symmetries of the model. The L^{-1} factor means that we will be considering subharmonics.

We will prove our results for frequencies of this type and obtain estimates which are rather uniform – in particular independent of L. This will allow to extend the results to $\omega \in \mathbb{R}^d$ by approximating it by rationals.

We denote by $\mathcal{R}_\omega$ the module

$$\mathcal{R}_\omega = \{k \in N\mathbb{Z}^d \mid \omega \cdot k = 0\}$$

where $\omega \cdot k$ denotes the usual inner product. We note that, when ω is rational, $\mathcal{R}_\omega$ is a $d-1$ dimensional module.

Given a module $\mathcal{R} \subset \mathbb{Z}^d$ we denote by $\mathcal{F}_\mathcal{R} = \mathbb{Z}^d/\mathcal{R}$ a fundamental domain of the translations in $\mathcal{R}$. If $\mathcal{R}$ is a $d-1$ dimensional module $\mathcal{F}_\mathcal{R}$ is isomorphic to $\mathbb{Z}_N^{d-1} \times \mathbb{Z}$ and can be considered as a discrete version of $\mathbb{R}^{d-1}/\mathcal{R} = \mathbb{T}^{d-1} \times \mathbb{R}$.

In the case of $\mathcal{R} = \mathcal{R}_\omega$ we will denote simply $\mathcal{F}_\omega$ rather than $\mathcal{F}_{\mathcal{R}_\omega}$. In the case of $\mathcal{R} = L\mathbb{Z}^d$, $L \in \mathbb{N}$, we will denote $\mathcal{F}_{L\mathbb{Z}^d}$ as $\mathcal{F}_L$. Note that with this notation $\mathcal{F}_N$ is just a fundamental domain for the system under the translations assumed to exists in *H1*.

If $\mathcal{R} = L\mathcal{R}_\omega$, $L \in \mathbb{N}$ we will denote $\mathcal{F}_{L\mathcal{R}_\omega} = \mathcal{F}_{L,\omega}$. The sets

$$\begin{aligned} \mathcal{F}_\omega^A \ &= \{i \in \mathcal{F}_\omega \mid 0 \leqslant \omega \cdot i \leqslant A|\omega|\} \\ \mathcal{F}_{L,\omega}^A &= \{i \in \mathcal{F}_{L,\omega} \mid 0 \leqslant \omega \cdot i \leqslant A|\omega|\} \end{aligned}$$

are finite sets. We note that $\mathcal{F}_\omega^A$, $\mathcal{F}_{L,\omega}^A$ are invariant under translations in $\mathcal{R}_\omega$ and $L\mathcal{R}_\omega$, respectively.

Again, we note that $\mathcal{F}_{L,\omega}^A$ is a covering – in the directions perpendicular to ω of $\mathcal{F}_L^A$.

The reason behind this notation is that, since we are considering periodic problems, it will be convenient to reduce ourselves to considering them as defined in the fundamental domain of the translations, $\mathcal{F}_{L,\omega}$. The fundamental theorem is, geometrically, the product of a finite surface and an unbounded direction, the direction of the normal ω. In order to study problems in compact domains, we introduce cut-off versions of the fundamental domain, $\mathcal{F}_{L,\omega}^A$. We will study the problem in the cut-off fundamental domain and remove the cut-off A by developing enough *a priori estimates*.

3.1.3. Symmetric configurations

Given a $\mathbb{Z}$-module $\mathcal{R}$ we denote by $\mathcal{P}_{\mathcal{R}}$ the set of configurations which are invariant under translations in $\mathcal{R}$

$$\mathcal{P}_{\mathcal{R}} = \left\{ s \in \mathcal{C} \;\middle|\; s_{i+k} = s_i \;\; \forall \; i \in \mathbb{Z}^d, \; k \in \mathcal{R} \right\}$$

In the case of $\mathcal{R} = \mathcal{R}_\omega$ we will denote $\mathcal{P}_{\mathcal{R}_\omega} = \mathcal{P}_\omega$. Similarly $\mathcal{P}_{L,\omega} = \mathcal{P}_{L\mathcal{R}_\omega}$.

We will also consider

$$\mathcal{P}^A_{L,\omega} = \left\{ s \in \mathcal{P}_{L,\omega} \;\middle|\; s_i = -1 \text{ when } \omega \cdot i > A|\omega|, \quad s_i = +1 \text{ when } \omega \cdot i < 0 \right\}$$

These classes of configurations consist of configurations which are periodic in the directions parallel to the plane and satisfy boundary conditions on the top and the bottom of the slab of width A parallel to the plane Π.

When $L = 1$ we will simply write $\mathcal{P}^A_\omega$.

Note that a configuration in $\mathcal{P}^A_{L,\omega}$ is determined when we prescribe it in the finite set $\mathcal{F}^A_{L,\omega}$. (We can determine for all the other points either by using the periodicity in the translations or by the boundary conditions.)

Note that the classes $\mathcal{P}_\omega$ above involve not only periodicity but also some boundary conditions. We have taken the convention that ω is oriented in the sense in which the conditions go from positive to negative. Of course, since we are considering ω an arbitrary vector, taking the opposite convention just amounts to changing ω into $-\omega$.

When the magnetic field is not present, it is easy to see that changing s into $-s$ does not change the energy, hence, all the results will be the same when we change ω into $-\omega$ nevertheless, when $h \neq 0$, in general, the results could change when ω changes into $-\omega$.

We will eventually take A to ∞ but, as it is well known in statistical mechanics some information about the boundary remains.

3.1.4. Operations on configurations

We introduce the notation

$$\begin{aligned} (s \wedge t)_i &= \min(s_i, t_i) \quad i \in \mathbb{Z}^d \\ (s \vee t)_i &= \max(s_i, t_i) \quad i \in \mathbb{Z}^d. \end{aligned}$$

Given any configurations s, t, there are non-negative functions α, β on the lattice so that we can write:

$$\begin{aligned} s &= s \wedge t + \alpha \\ t &= s \wedge t + \beta \\ s \vee t &= s \wedge t + \alpha + \beta \end{aligned} \tag{9}$$

Note that $s + t = s \vee t + s \wedge t$. We also note that if $s, t \in \mathcal{P}^A_{L,\omega}$, then $s \wedge t$, $s \vee t \in \mathcal{P}^A_{L,\omega}$.

Remark 3.1. In comparing with ref. 4 it is useful to observe that if we use the description of configurations by sets as in (4), we have

$$\begin{aligned} \mathcal{S}(s \vee t) &= \mathcal{S}(s) \cap \mathcal{S}(t) \\ \mathcal{S}(s \wedge t) &= \mathcal{S}(s) \cup \mathcal{S}(t) \end{aligned} \tag{10}$$

3.2. Minimizers and infimal minimizers

Now, we turn our attention to the problem of producing minimizers in spaces of periodic configurations. The goal of this section is to produce a minimizer that enjoys some remarkable properties.

We call attention to the fact that the results of this section work under the assumptions of weak ferromagnetism and do not require the fact that the interaction is non-degenerate.

Since configurations on $\mathcal{P}^A_{L,\omega}$ are determined by the values on a the finite set $\mathcal{F}^A_{L,\omega}$, it is natural to consider a reduced energy adapted to the periodic problem,

$$H_{\mathcal{F}^A_{L,\omega}}(s) = \sum_{i,j \in \mathcal{S}} J_{ij}(s_i s_j - 1) + \sum_{i \in \Gamma} h_i s_i.$$

The sum is extended so that we count all the bonds twice. This is somewhat cumbersome to write since due to the periodicity of the configurations, we have to take special provisions for the bonds that jump across the boundary. The boundary corresponding to the constraint $\omega \cdot i = A|\omega|$ has to be written slightly differently.

Since $\mathcal{P}^A_{L,\omega}$ is finite it is clear that $H_{\mathcal{F}^A_{L,\omega}}$ reaches its minimum on this set. It can, of course, well happen that there are several configurations which achieve the minimum.

Note that the minimizers of the periodified (and constrained) problem thus produced, minimize the functional $\mathcal{F}^A_{L,\omega}$ among configurations in $\mathcal{P}^A_{L,\omega}$. Therefore, they satisfy (2.1) but with two important caveats: the sets Γ that are at a distance bigger than R from the cut-off. That is: $\Gamma \subset \{i \in \mathbb{Z}^d \,|\, R|\omega| \leqslant \omega \cdot i \leqslant (A-R)|\omega|\}$. We also require that the test configurations u in (2.1) are configurations in $\mathcal{P}^A_{L,\omega}$.

At this state of the argument, there is no reason why they should be minimizers with respect to more general perturbations that have less periodicity or that violate the other constraints. Hence, the minimizers of the reduced problem could fail to be ground states. This is, a manifestation of symmetry breaking. Also, as it can be seen in examples in Section 5, there could be minimizers that oscillate widely (their oscillation is proportional to L).

Hence, we will select a particular minimizer (infimal minimizer) that enjoys special properties. The infimal minimizer will be selected roughly as the minimizer which stays closer to the lower constraint and, hence, oscillates the least. We will show that this infimal minimizer does not experience symmetry breaking and that enjoys a property analogous to the property called Birkhoff property in dynamical systems.

A good deal of the argument later will be precisely showing that there is no symmetry breaking for the infimal minimizer. This will have as a consequence that all the minimizers remain as minimizers under multiplication of the period. This is not completely obvious because, as we will see there are more minimizers when we increase the period. The proof of this absence of symmetry breaking will require the use of the ferromagnetism assumption, which we have not used so far.

We hope that the examples in Section 5 will clarify this situation.

The construction of the infimal minimizer and the proof of the fact that it has no symmetry braking require that we use the assumptions we have made on the structure of the functional and on the ferromagnetism of the interaction.

We start by observing that the functional H defining the models has a quadratic part, a linear part, and a constant. Namely:

$$Q_\Gamma(s) = \sum_{\substack{i\in\Gamma \\ j\in\mathbb{Z}^d \\ |i-j|\leqslant R}} J_{ij} s_i s_j$$

$$L_\Gamma(s) = \sum_{i\in\Gamma} h_i s_i$$
$$C_\Gamma = \sum_{i\in\Gamma} 1$$

We will also introduce the notation

$$Q_\Gamma(s,t) = \sum_{\substack{i\in\Gamma\\ j\in\mathbb{Z}^d\\ |i-j|\leqslant R}} J_{i,j} s_i t_j$$

so that $Q_\Gamma(s) = Q_\gamma(s,s)$.

Similar definitions hold when we restrict the sum to a finite set, e.g. to consider the Hamiltonian in a set or when we consider periodified problems.

With the notations above, we have the following identity

$$H_\Gamma(s\wedge t) + H(s\vee t) = H_\Gamma(s) + H_\Gamma(t) + Q_\Gamma(\alpha,\beta) \tag{11}$$

where α, β are given in (9).

Note that (11) remains also true when we restrict the sum to some particular sets of bonds.

Under the hypothesis of ferromagnetism, for all $\alpha, \beta \geqslant 0$ we have:

$$Q_\Gamma(\alpha,\beta) \leqslant 0 \tag{12}$$

because $\alpha_i\beta_i \geqslant 0,\ J_{ij} \leqslant 0$.

Therefore we have:

Proposition 3.1. If s, t are minimizers of $H_{\mathcal{F}^A_{L,\omega}}$ in $\mathcal{P}^A_{L,\omega}$, then so are $s\vee t$, $s\wedge t$. In particular, there is an infimal minimizer defined by:

$$s^A_{L,\omega} = \bigwedge_{s\in Minimizers} s \tag{13}$$

Proof. Note that $s\vee t$, $s\wedge t$ are configurations with the same periodicity as s, t. hence, by s, t being minimizers, we have

$$H_\Gamma(s\wedge t) \geqslant H_\Gamma(s) = H_\Gamma(t)$$
$$H_\Gamma(s\vee t) \geqslant H_\Gamma(s) = H_\Gamma(t)$$

On the other hand, using (11) and (12), we have:

$$H_\Gamma(s \wedge t) + H_\Gamma(s \vee t) \leqslant H_\Gamma(s) + H_\Gamma(t)$$

Therefore,

$$H_\Gamma(s \wedge t) = H_\Gamma(s \vee t) = H_\Gamma(s) + H_\Gamma(t)$$

and $s \wedge t$, $s \vee t$ are minimizers. ∎

Clearly, once we prescribe $\mathcal{R}$, A, the infimal minimizer is unique since it is given by the formula (13).

This has the important consequence that there is no symmetry breaking (Proposition 3.2) which in turn will lead to the fact that S_ω^A is a minimizer against configurations that respect the boundary conditions (Proposition 3.3).

Remark 3.2. The physical interpretation of the infimal minimizer is that it would be the minimizer if we introduced a very small magnetic field (or an small pressure in the lattice gas interpretation) but maintained the lower constraint. Hence, it can be compared with the use of "*infinitesimal fields*", which is very common in the physical literature.

3.2.1. Absence of symmetry breaking

In the following proposition, we show that for any $K \in \mathbb{N}$, if we consider perturbations with K-times the period, the infimal minimizer is also a minimizer among those. Indeed, it is the infimal minimizer for functions with K period.

Proposition 3.2. Let $K, M \in \mathbb{N}$. Denote $L = K \cdot M$. Let $A \in \mathbb{R}^+$. Then

$$s_{L,\omega}^A = s_{M,\omega}^A \tag{14}$$

Proof. We define

$$\widetilde{s} = \bigwedge_{k \in M\mathcal{R}_\omega / L\mathcal{R}_\omega} \mathcal{T}_k s_{L,\omega}^A$$

since $0 \in M\mathcal{R}_\omega / L\mathcal{R}_\omega$ we have $\widetilde{s} \leqslant s_{L,\omega}^A$.

It is important to note that $\widetilde{s} \in \mathcal{P}_{M,\omega}^A$.

Since $\mathcal{T}_k$, $s_{L,\omega}^A$ are minimizers in $\mathcal{P}_{L,\omega}^A$, we obtain, applying Proposition 3.3, that $\widetilde{s}$ is a minimizers in $\mathcal{P}_{L,\omega}^A$.

From the definition of infimal minimizer we obtain

$$\widetilde{s} \geqslant s^A_{L,\omega}$$

which with the observation after the definition implies $\widetilde{s} = s^A_{L,\omega}$.

Using that $s^A_{L,\omega}$ and $s^A_{M,\omega}$ are minimizers of their respective functionals we obtain

$$H_{\mathcal{F}^A_{L,\omega}}(s^A_{L,\omega}) \leqslant H_{\mathcal{F}^A_{L,\omega}}(s^A_{M,\omega})$$
$$H_{\mathcal{F}^A_{M,\omega}}(s^A_{M,\omega}) \leqslant H_{\mathcal{F}^A_{M,\omega}}(s^A_{L,\omega})$$

On the other hand, for configurations $s \in \mathcal{P}^A_{M,\omega}$ we have

$$H_{\mathcal{F}^A_{L,\omega}}(s) = \#\left(\frac{M\mathcal{R}_\omega}{L\mathcal{R}_\omega}\right) H_{\mathcal{F}^A_{L,\omega}}(s) \tag{15}$$

Using (18) and (15) we obtain

$$H_{\mathcal{F}^A_{L,\omega}}(s^A_{L,\omega}) = H_{\mathcal{F}^A_{L,\omega}}(s^A_{M,\omega})$$
$$H_{\mathcal{F}^A_{M,\omega}}(s^A_{L,\omega}) = H_{\mathcal{F}^A_{M,\omega}}(s^A_{M,\omega})$$

Hence we obtain that $s^A_{L,\omega}$ is a minimizer in $\mathcal{P}^A_{M,\omega}$ and $s^A_{M,\omega}$ is a minimizer in $\mathcal{P}^A_{L,\omega}$.

Therefore, using the definition of infimal minimizer, we obtain

$$s^A_{L,\omega} \geqslant s^A_{M,\omega}$$
$$s^A_{M,\omega} \geqslant s^A_{L,\omega}$$

and therefore, the claim of Proposition 3.2. ∎

As a corollary of Proposition 3.2, we obtain:

Corollary 3.1. All minimizers in $\mathcal{P}^A_{\mathcal{R}_\omega}$ are minimizers in $\mathcal{P}^A_{K\mathcal{R}_\omega}$

The proof is simply observing that the energy of a minimizer with a certain period is the same as that of the infimal minimizer.

Hence, if we consider a minimizer u with unit period its energy in the unit period will be the same as that of the infimal minimizer of unit period. Since the infimal minimizer of period K is just K^d copies of the infimal minimizer, we obtain that the energy of the minimizers with period K is K^d times the energy of a minimizer of period 1, which is the same as the energy of considering u in period K. Hence, u is also a minimizer in period K. ∎

Remark 3.3. The phenomenon that minimizers under perturbations of one period are not minimizers under perturbations of a longer period – hence the energy of the minimizer decreases with the period – happens in many variational problems. It appears already in ref. 9.

This phenomenon often prevents to take the limit of minimizers when we change the period to an irrational period.

Note that the argument above implies that if there is a way of selecting a unique minimizer, the Hedlund phenomena does not happen.

The Corollary 3.1 is somewhat surprising since we will see in Section 5 that, even in the classical Ising model, there are more minimizers in $\mathcal{P}^A_{K\mathcal{R}_\omega}$ than in $\mathcal{P}^A_{\mathcal{R}_\omega}$.

Notice that, since K is arbitrary, it immediately follows from Proposition 3.2. given any perturbation of $s^A_{L,\omega}$ of bounded support, we can find a K large enough so that it can be considered as a perturbation in a fundamental domain of the KN perturbation. Hence, we have established:

Proposition 3.3. $s^A_{L,\omega}$ is a class-A minimizer among the configurations in $\mathcal{P}^A_\omega$.

That is, s^A_ω is a minimizer for all the functions that satisfy the boundary conditions, irrespective of periodicity. Given the fact that $S^A_{L,\omega}$ is independent of L we will just use the notation S^A_ω from now on.

3.2.2. The Birkhoff property

The following property of the infimal minimizer is quite analogous to a property that is commonly called "Birkhoff property" in dynamical systems.

In the following Proposition 3.4, we prove it for the infimal minimizer.

Proposition 3.4. Let s^A_ω be the infimal minimizer as before (in particular, recall that $\omega \in \frac{1}{N}\mathbb{Z}^d$)

Let $k \in N\mathbb{Z}^d$ then,

$$\begin{aligned} \mathcal{T}_k s^A_\omega &\leqslant s^A_\omega \quad k\cdot\omega \leqslant 0 \\ \mathcal{T}_k s^A_\omega &\geqslant s^A_\omega \quad k\cdot\omega \geqslant 0 \end{aligned} \tag{16}$$

Proof. Because of (8) $\mathcal{T}_k s_\omega^A$ is a minimizer for $H_{\mathcal{T}_k \mathcal{F}_\omega^A}$. We note that $\mathcal{T}_k s_\omega^A \in \mathcal{P}_\omega^A$.

We will prove the inequality (16) for

$$\omega \cdot k \leqslant 0. \tag{17}$$

The other case will be identical.

Note that, under the assumption (17), $i \cdot \omega \leqslant 0$ implies $(i+k) \cdot \omega \leqslant 0$. Hence,

$$(\mathcal{T}_k s_\omega^A)_i = (s_\omega^A)_{i+k} = +1.$$

Therefore,

$$s_\omega^A \wedge \mathcal{T}_k s_\omega^A \in \mathcal{P}_\omega^A.$$

Similarly, we obtain that

$$s_\omega^A \vee \mathcal{T}_k s_\omega^A \subset \mathcal{T}_k \mathcal{P}_\omega^A$$

We have therefore

$$\begin{aligned} H_{\mathcal{F}_\omega^A}(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) &\geqslant H_{\mathcal{F}_\omega^A}(s_\omega^A) \\ H_{\mathcal{T}_k \mathcal{F}_\omega^A}(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) &\geqslant H_{\mathcal{T}_k \mathcal{F}_\omega^A}(\mathcal{T}_k s_\omega^A) \end{aligned} \tag{18}$$

We note that

$$\begin{aligned} \mathcal{F}_\omega^A &\subset \mathcal{F}_\omega^{A-k\cdot\omega/|\omega|} \\ \mathcal{T}_k \mathcal{F}_\omega^A &\subset \mathcal{F}_\omega^{A-k\cdot\omega/|\omega|} \end{aligned}$$

Moreover, denoting $\Gamma \equiv \mathcal{F}_\omega^{A-k\cdot\omega/|\omega|}$, the periodicity and the zero flux condition imply that

$$\begin{aligned} H_\Gamma(s) &= H_{\mathcal{F}_\omega^A}(s) \quad \forall\ s \in \mathcal{P}_\omega^A \\ H_\Gamma(s) &= H_{\mathcal{T}_k \mathcal{F}_\omega^A}(s) \quad \forall\ s \in \mathcal{T}_k \mathcal{P}_\omega^A \end{aligned} \tag{19}$$

The reason for this equality is that in $\Gamma - \mathcal{F}_\omega^A$, because of the boundary conditions, the quadratic interaction term does not give any contribution. The contribution of the magnetic field term is zero because of the zero flux condition. Hence (18) becomes:

$$H_\Gamma(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) \geqslant H_\Gamma(s_\omega^A)$$
$$H_\Gamma(s_\omega^A \vee \mathcal{T}_k s_\omega^A) \geqslant H_\Gamma(\mathcal{T}_k s_\omega^A)$$

Using (12), we obtain:

$$H_\Gamma(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) \geqslant H_\Gamma(s_\omega^A),$$
$$H_\Gamma(s_\omega^A \vee \mathcal{T}_k s_\omega^A) \geqslant H_\Gamma(s_\omega^A)$$

Using again (19) we obtain:

$$H_{\mathcal{F}_\omega^A}(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) \geqslant H_{\mathcal{F}_\omega^A}(s_\omega^A)$$

Therefore $s_\omega^A \wedge \mathcal{T}_k s_\omega^A$ is a minimizer. Since s_ω^A is the infimal minimizer, we obtain

$$s_\omega^A \wedge \mathcal{T}_k s_\omega^A \geqslant s_\omega^A$$

Therefore

$$\mathcal{T}_k s_\omega^A \geqslant s_\omega^A$$

which is the desired conclusion.

The case $\omega \cdot k \leqslant 0$ is proved exactly in the same way. ∎

Remark 3.4. We note that Propositions 12 and 3.4 have a natural geometric interpretation in terms of perimeters of contours. For example, the conclusion Proposition 12 reads:

$$\mathrm{Per}(\mathcal{S}_1 \cup \mathcal{S}_2) + \mathrm{Per}(\mathcal{S}_1 \cap \mathcal{S}_2) \leqslant \mathrm{Per}(\mathcal{S}_1) + \mathrm{Per}(\mathcal{S}_2)$$

Such interpretations appear naturally in the geometric measure theory problems considered in ref. 4.

3.3. Bounding the oscillation of the infimal minimizer

To finish the proof of Theorem 2.1, we will just need to show that, if we take A large enough – but independent of ω, the orientation of the interface –, the infimal minimizer will be an unconstrained minimizer and will not touch the boundaries.

The basic idea is that a minimizer cannot oscillate too much in an small scale since this will force it to have a very large energy in this scale and one can easily produce configurations with smaller energy. Using the Birkhoff property, we will use this information on small scales to control the large scales.

More precisely, our goal is to show.

Lemma 3.1. There exists an M large enough – depending on the properties of the model but independent of ω – such that for any $A \geqslant M$, we have:

$$s_\omega^A = s_\omega^M$$

Lemma 3.1 has as a corollary that the infimal minimizer s_ω^M is completely unconstrained.

Indeed, if there was a periodic configuration u such that $H(u) \leqslant H(s)$ and $\partial u \subset \{i \mid -A|\omega| \leqslant i \cdot \omega \leqslant A|\omega|\}$, we see that for some $k \in \mathbb{Z}^d$ we have $\partial \mathcal{T}_k u \subset \{i \mid -A|\omega| \leqslant i \cdot \omega \leqslant A|\omega|\}$.

Hence,

$$H(u) \leqslant H(s_\omega^{2A}) = H(s_\omega^M).$$

In other words, the energy of the configuration s_ω^M cannot be lowered by compact perturbations. ∎

Remark 3.5. The above corollary can be interpreted as a manifestation of the *"action-reaction"* principle. Once we know that the upper constraint is not acting on the interface, we conclude that the lower constraint is not acting either.

Once we have that s_ω^M is a ground state and that its interface is contained in a strip of width independent of ω, we see that, given $\omega^* = \lim_{n\to\infty} \omega_n$ with $\omega_n \in \mathbb{Q}^d$ we can – by passing to a subsequence – obtain $s_{\omega^*} = \lim s_{\omega_n}$. This s_{ω^*} will be a ground state and therefore, we have established Theorem 2.1 as soon as we prove Lemma 3.1.

The rest of this section is devoted to proving Lemma 3.1.

We introduce the notation for $\ell \in \mathbb{N}$, $x \in \mathbb{Z}^d$

$$\mathcal{C}_x^\ell \equiv \{0, \dots, \ell - 1\}^d + x.$$

That is $\mathcal{C}_x^\ell$ is a cube of side ℓ with the lower vertex at x.

A corollary of Proposition 3.4 is:

Proposition 3.5. If $(s_\omega^A)_i = -1$ for all $i \in \mathcal{C}_x^N$, then,

$$(s_\omega^A)_i = -1 \quad \forall\ i \mid | \omega \cdot i \geqslant x \cdot \omega + N\sqrt{d} \cdot |\omega|.$$

Proof. By the Birkhoff property

$$(s_\omega^A)_i = -1\ \forall\ i \in \bigcup_{\substack{k \in N\mathbb{Z}^d \\ k \cdot \omega \geqslant 0}} \mathcal{C}_{x+k}^N,$$

The set above is a collection of cubes of size N on a semi-lattice of size N. Hence, it contains a semi-plane. ∎

In view of Proposition 3.5 to show that the interfaces of the infimal ground state s_ω^A is contained in a strip of uniform width M it suffices to show:

Proposition 3.6. Assume that $A \geqslant M$ (where M can be chosen independent of ω) then there exists an $x \in \mathbb{Z}^d$ such that

$$\begin{gathered} 0 \leqslant \omega \cdot x \leqslant A - \sqrt{d}N \\ (s_\omega^A)_i = -1 \quad i \in \mathcal{C}_x^N. \end{gathered}$$

Proof. This will be a covering argument very similar to that used in ref. 4 but somewhat simpler since the density estimates used in ref. 4 are not needed in this case.

We will show that, we can bound the energy of a configuration from below by the number of cubes it touches multiplied by a constant. We also note that the energy of a configuration is bounded by the energy of a plane, which can be bounded from below by the area of the base of the strip times a constant. Moreover, the number of cubes in a strip is proportional to the area of the base of the strip multiplied by the height of the strip (see Fig. 1). The upshot of the discussion is that if the width of the strip is large enough (independent of the orientation), then there has to be a unit cube that is not touched by the interface. In the following we give a more formal proof.

Given a fundamental domain $\mathcal{F}_\omega^A$ we consider a collection of disjoint cubes centered in points x

$$\mathcal{C}_x^{3N}, \quad x \in 3N\mathbb{Z}^d, \quad \mathcal{C}_x^{3N} \subset \mathcal{F}_\omega^A$$

In each of the cubes $\mathcal{C}_x^{3N}$, we consider the cubes $\mathcal{C}_x^N$ with the same center x than $\mathcal{C}_x^{3N}$ but well inside $\mathcal{C}_x^{3N}$. We note that the cubes $\mathcal{C}_x^{3N}$ do not overlap and cover the fundamental domain rather completely except for a sliver near the edges.

We make several observations. The first two are purely geometric about the covering as indicated. The next two involve the Hamiltonians and the properties of the ground states. Note that item (iii) below uses the full strength of the assumption H2, so far, we have used only the weak ferromagnetic part.

We can find a constant B (geometrically the area of the base of $\mathcal{F}_\omega^A$) such that

(i) Denote by $\mathcal{B}$ the set

$$\mathcal{B} \equiv \mathcal{F}_\omega^A - \bigcup_{x\in\Sigma} \mathcal{C}_x^{3N}$$

(the set that is not covered by the cubes).
We have

$$\# \ \mathcal{B} < B\alpha$$

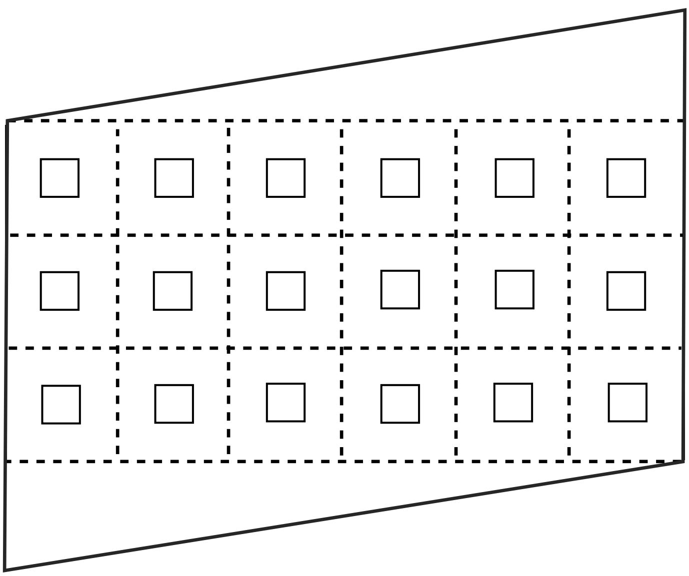

Fig. 1. Illustration of the fundamental domain $\mathcal{F}_\omega^A$, the cubes $\mathcal{C}_x^{3N}$ and the cubes $\mathcal{C}_x^N$ used in the proof of Proposition 3.6.

where $\#\mathcal{B}$ denotes the number of sites in $\mathcal{B}$.
This means that we can cover the whole fundamental domain by the cubes $\mathcal{C}_x^{3N}$ except for a thin sliver near the boundary.
The usual formula for the volume shows that it suffices to take $\alpha = \sqrt{d}\, 3N$.

(ii) Given M, we have that

$$\#\left\{x \mid d(\mathcal{C}_x^{3N}, \Pi) \leqslant M\right\} \geqslant BM \frac{1}{(3N)^d} - \frac{B\alpha}{(3N)^d}.$$

Once we have item (i) this result follows simply by noticing that each center has associated a cube of volume $(3N)^d$. So that the number of centers has to be bigger or equal than the total volume covered divided by the volume of each cube.

(iii) Given any configuration s we have

$$H_{\mathcal{C}_x^{3N}}(s) \geqslant 0 \qquad H_{\mathcal{C}_{y(x)}^{N}}(s) \geqslant 0.$$

If the cubes do not involve any interfaces, the result is obvious because we assumed that the flux of the magnetic field is zero, so that the final result is zero.
If there is an interface, by assumption H2 there is one interaction term which is negative and bounded away from zero and the other interaction terms are positive. The other contribution to the energy is the the magnetic field over the incomplete box. By assumption H4, which says that the magnetic field is small enough, these terms cannot overcome the negative term which was bounded away from zero.

(iv) In this term we make more precise the results before when there is an interface in the small cube.
Assume that

$$\partial s \cap \mathcal{C}_{y(x)}^{N} \neq \emptyset.$$

Then

$$H_{\mathcal{C}_x^{3M}}(s) \geqslant \gamma.$$

Observe that it suffices to take

$$\gamma = \inf_{|i-j|=1} J_{ij} - 3^d \Sigma |h_i|.$$

(v) Finally, we obtain a bound of the energy associated to the set $\mathcal{B}$ introduced in point (i) which is not covered by the cubes.

$$H_{\mathcal{B}}(s) \geqslant -B\alpha \sup_i |h_i|.$$

This is obvious because of the point (i) and the interaction can be bounded from below by the magnetic field terms, which can be bounded from below as indicated.

Note that, under the assumption that h is small we have that the constant γ is strictly positive.

In the previous remarks we have obtained a lower bound of the number of cubes (see item (ii)). Note that this number grows with M and it is proportional to B.

We also obtained a lower bound of the energy of the cubes $\mathcal{C}_x^{3N}$ for which $\mathcal{C}_x^N$ intersects the interface (see item (iv)). This and item (iii) lead to a lower bound for the energy in terms of the number of cubes $\mathcal{C}_x^N$ which intersect the interface. Since the energy of the minimizer is bounded from above by a number independent of M but proportional to B. – by comparing e.g. with a plane –. Putting bounds together the upper and lower bounds for the energy of the interface, we obtain an upper bound on the number of cubes $\mathcal{C}_x^N$ which intersect the interface.

By comparing the upper bound on the number of cubes that intersect the interface and the lower bounds in the number of existing cubes, we conclude that, when M is large enough there is a cube that does not intersect the interface.

We proceed to give some more details on the argument which will allow us to check that the width required is indeed independent on the orientation of the plane.

Proposition 3.7. Denote by $\mathcal{N}(s)^M$ the number of cubes $\mathcal{C}_{y(x)}^N$ which intersect the interface of s and such that $d(y(x),\pi) \leqslant M$. Then, we have for $A \geqslant M$

$$H_{\mathcal{F}_\omega^A}(s) \geqslant \mathcal{N}^M(s)\gamma - B\alpha$$

The proof of Proposition 3.7 is obvious if we realize that

$$H_{\mathcal{F}_\omega^A}(s) = H_{\mathcal{B}}(s) + \sum_{x\in\Sigma} H_{\mathcal{C}_x^{3M}}(s)$$

We note that all the $H_x^{3N}(s) \geqslant 0$. Hence, we obtain a lower bound of the sum if we restrict it only to the cubes such that a $\mathcal{C}_{y(x)}^N$ intersects the interface and $d(\mathcal{C}_s^{3N},\pi) \leqslant M$. Moreover, a lower bound of the term $H_{\mathcal{B}}(s)$ is contained in the point (v). ∎

We also observe that the test configuration s^* defined by

$$s_i^* = \begin{cases} +1 & \omega \cdot i \leqslant |\omega| \\ -1 & \text{otherwise} \end{cases}$$

satisfies

$$H_{\mathcal{F}_\omega^A}(s^*) \leqslant B\delta$$

where $\delta \leqslant \sup |J_{ij}| + \sup h_i$. Therefore:

$$H_{\mathcal{F}_\omega^A}(s_\omega^A) \leqslant B\delta \tag{20}$$

Comparing (20) with 3.7 we obtain

$$\mathcal{N}^M(s_\omega^A) \leqslant \frac{B\delta}{\gamma} + \frac{B\alpha}{\gamma}$$

Since the number of cubes at a distance M is bounded from below in the point (ii), we obtain that if

$$M \geqslant (3N)^d \left[(1-\alpha)^{-1} \left[\left(\frac{\delta + \alpha}{\gamma} \right) + 1 \right] \right] \tag{21}$$

there is one cube at a distance less than M such that it does not intersect the interfaced of s_ω^A. ∎

We emphasize that the condition (21) is independent of B and, hence, independent of ω.

Applying Proposition 3.5 with Proposition 3.6 we obtain that $(s_\omega^A)_i = -1$ whenever $\omega \cdot i \geqslant M|\omega|$ independently of A. This establishes Lemma 3.1 and, by the arguments at the beginning of this section, it proves Theorem 2.1.

4. PROOF OF THEOREM 2.2

4.1. Existence of the limits

We will first prove the existence of the limit of the average energy when we consider sequences of cubes.

Once we prove the result with enough uniformity with respect to the direction and with respect to the ground state, as well as with very explicit error estimates, the existence of the limits claimed in Theorem 2.2 will follow easily by approximating the domain $\lambda\Sigma$ by cubes.

The first result that we will prove that the average energy of a large cube is largely independent of which cube and which plane-like minimizer we are considering.

This will be the basis of much of the uniformity that we need later. Note that we establish that for cubes of size L, up to errors which are much smaller than the area of the boundary, the energy associated to the cube is determined by the area of the intersection.

Proposition 4.1. There exists a constant Ω independent of the cubes, the strips and the ground states (it may depend on the model and the constant M in Eq. (5) for the state) with the following property:

Let s, s' be class-A minimizers, contained in strips Γ, Γ' of width M around parallel planes Π, Π' respectively.

Assume without loss of generality that $\Gamma + k = \Gamma'$ for some $k \in N\mathbb{Z}^d$.

Let Q, Q' be cubes of side L – L sufficiently large – Assume that

$$|\#(\Gamma \cap Q) - \#(\Gamma' \cap Q')| \leqslant (\Omega/2) L^{d-2} \tag{22}$$

Then,

$$|H_Q(s) - H_{Q'}(s')| \leqslant \Omega L^{d-2} \tag{23}$$

Note that in Theorem 2.1 we have shown that the constant M can be taken to be independent of the orientation for the infimal minimizer. Hence, if we apply Proposition 4.1 to the configurations produced in Theorem 2.1, we get that Ω depends only on the model. Of course, the way that we formulated it, applies to other ground states provided that they are plane-like.

The assumption that the strips are congruent under translations can always be arranged by making them slightly bigger. (so that the interfaces will always be contained) anyway. The amount is not bigger than $N\sqrt{d}$. Hence, for large L this is rather irrelevant.

Proof. The proof is very simple in the case that the cubes and the intersections are congruent by translations which are multiple of N, the period of the interaction. We can produce an configuration s'' that agrees with s outside of Q and whose intersection with Q is a translation by multiples of N of the intersection of s' with Q.

Since s is a ground state, we conclude that $H_{Q^R}(s'') \geqslant H_{Q^R}(s)$. But $|H_{Q^R}(s'') - H_{Q'^R}(s')| \leqslant \Omega L^{d-2}$ because the terms in the energy differ only in the boundary terms. Since the interface is contained in a strip of width M, the number of affected terms can be bounded by $CMRL^{d-1}$ where C

is a constant that depends only on the dimension and the geometry and R is the range of the interaction.

By exchanging the role of s, s', we obtain the desired result.

When the cubes are not congruent by translations multiples of N, we note that we can discard some points in the cubes, which are at a distance not more than N from the boundary so that we obtain cubes $\tilde{Q}$, $\tilde{Q}'$ that are congruent under translations by N.

Clearly, we have $|H_Q(s) - H_{\tilde{Q}}(s)| \leqslant \Omega L^{d-2}$. ∎

In view of Proposition 23, from now on, we will speak about the energy of a plane-like ground state in a cube of length L and we will not bother specifying which cube or which ground state. As Proposition 4.1 shows this is defined up to an additive term of size $\leqslant \Omega L^{d-2}$, which will not affect any of the subsequent arguments.

The following result gives us some crude bounds of a form similar to that of the desired limit. Later we will refine them.

Proposition 4.2. Under the assumptions of Theorem 2.2.

Let s be a plane-like minimizer. Let Q be a cube of length L.

For some suitable constants Ω_1,Ω_2, Ω_3 depending only on the model, and on M, we have:

$$\begin{aligned}\Omega_1 |\Pi_\omega \cap Q|_{d-1} - \Omega_3 L^{d-2} &\leqslant H_Q(s) \\ &\leqslant \Omega_2 |\Pi_\omega \cap Q|_{d-1} + \Omega_3 L^{d-2}. \qquad (24)\end{aligned}$$

Proof. The upper bound is obtained by comparing the energy of the state s with the state that has an interface along the plane. The terms L^{d-2} come from the modifications that one has to do to match the boundary conditions.

The lower bound follows from noting that the interface is the boundary of a set, so that we can bound the number of points in the interface by the area of the intersection. The arguments are very similar to the remarks that lead to a proof of Proposition 3.7. We refer there for more details.

The energy of interaction of a site in the boundary is bounded from below by a constant. Hence, the energy of the interaction is bounded from below by a constant times the number of points in the interface. Hence, by a constant times the area of the intersection of the plane with the cube.

By the assumption of zero magnetic flux, the absolute value of the energy due to the magnetic field can be bounded by the strength of the magnetic field times the number of N-cubes that contain the some point in the interface. ∎

The following definition will be useful since it selects a particular class of intersections.

Definition 4.1. Given a cube and a strip, we say that the intersection with the cube is clean if

- Whenever the intersection with one face of the cube is non-empty, the intersection with the parallel face of the cube is not empty.

- The intersection does not include any intersection of more than two faces.

Note that for all the clean intersections between cubes of the same length and parallel planes have the same area.

Now, we study the limit of the cubes growing larger.

Proposition 4.3. Let s be a plane-like minimizer. Let Q_L, Q_{2L} be cubes of size L, $2L$, respectively.

Assume that the plane-like minimizer s intersects cleanly Q_L and that the minimizer s' intersects cleanly Q_{2L}.

Then

$$|2^{d-1}H_{Q_L}(s) - H_{Q_{2L}}(s')| \leqslant \Omega L^{d-2}. \tag{25}$$

Proof. Given the uniformity properties proved in Proposition 4.1, it suffices to observe that the intersection in the cube Q_{2L} can be covered by 2^{d-1} disjoint cubes with a clean intersection.

In effect, suppose without loss of generality that the plane of intersection is a graph over of a linear function of the first $d-1$ variables to the d one and that the angle with the horizontal is smaller than 1. (It suffices to reorder the components so that the d component is the largest one).

Take a dyadic decomposition of the base of Q_{2L}. For each of these $d-1$ cubes $\tilde{Q}$ of size L, we can find an interval I of size L so that the cube $\tilde{Q} \times I$ has a clean intersection with the plane. ∎

We define

$$\begin{aligned} \mathcal{A}^{+}(L) &= \sup L^{-d+1} H_{Q_L}(s), \\ \mathcal{A}^{-}(L) &= \inf L^{-d+1} H_{Q_L}(s) \end{aligned} \tag{26}$$

where the $\sup, \inf$ are taken over all the cubes of size L and all the plane like s that have a clean intersection with them.

Proposition 4.2 tells us that the functions $\mathcal{A}^{\pm}$ are well defined and that we have

$$\mathcal{A}^{+}(L)-\mathcal{A}^{-}(L)\leqslant \Omega L^{-1}.$$

Using Proposition 4.3 we have

$$|\mathcal{A}^{\pm}(2L)-\mathcal{A}^{\pm}(L)|\leqslant \Omega L^{-1}.$$

From this, it clearly follows that $\lim_{L\to\infty}\mathcal{A}^{\pm}(L)$ exists and that it is equal for both functions.

Moreover, the convergence is rather uniform.

If we approximate the set $\lambda\Sigma$ by cubes of size $\lambda^{1/2}$ we see that we can cover the intersection of $\lambda\Sigma\cap\Pi_\omega$ except for a set whose measure can be bounded by $\lambda^{d-2}\lambda^{1/2}$.

We have a number of cubes, each of which has an average energy $\mathcal{A}(\omega)$ up to an error $\lambda^{-1/2}$.

Hence, the desired result follows.

It seems that, if one used coverings more efficient than the covering by uniform cubes, one could get better estimates for the remainder, but we will not pursue this here. ∎

4.2. Convexity properties of the averaged energy

To prove the convexity of the averaged energy, the argument used in ref. 4 works without modification. For the convenience of the reader we repeat here the most salient steps. The argument is illustrated in Fig. 2 which is reproduced from ref. 4.

Given the uniformity properties established in the previous subsection, we can compute approximations of $\mathcal{A}(\omega)$ just by taking a very large set and computing the energy of the intersection of this set with any of the plane-like ground states whose interface lies in a neighborhood of the plane Π_ω

By the homogeneity, it is enough to show that

$$\mathcal{A}(\omega_1)+\mathcal{A}(\omega_2)\geqslant \mathcal{A}(\omega_1+\omega_2).$$

There is only anything to prove in the case that ω_1 is not parallel to ω_2.

By the uniformity of the limits, it is enough to take very large sets. We just take very large cylinders sets whose transversal section is indicated in Fig. 2. We see that taking the joining of the sets corresponding to ω_1 and ω_2 as comparisons with the infimal minimizer corresponding to ω_3 and noting that for all of them, the error from the average is uniformly small if the size is big enough, we obtain the desired result.

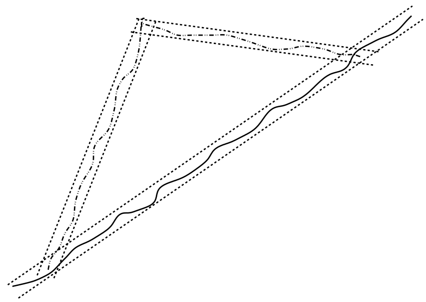

Fig. 2. Illustration of the argument to show that $\mathcal{A}$ is convex.

Since $\mathcal{A}$ is sublinear, it follows that it is Lipschitz. As we will see in Section 5 for the Ising model, it is not C^1.

5. SOME EXAMPLES

5.1. The classical Ising model

This corresponds to taking $J_{ij} = 1$ when $|i - j| = 1$ and 0 otherwise. In particular, this satisfies the very strong non-degeneracy assumption alluded to in Remark 2.4.

It is easy to see that the minimization problem in a periodic class admits minimizers that are not Birkhoff. For dimension $d = 2$, some of them are depicted in Fig. 3.

Non-Birkhoff minimizers can be constructed by fixing two points in the interface as required by the periodicity. The interface consists of a path that joins these two points and consists of a horizontal segment and a vertical segment. (The fact that these are minimizers is obvious because if we consider the interface as a path, the length is just the taxicab distance.)

It is clear that if we multiply by K the periodicity allowed in the configurations, a similar construction will give an interface that recedes from the plane by an amount K times larger. Hence, in the classical Ising model, there is symmetry breaking for the ground states.

Note that in any dimension, including $d = 2$, given a box of size K, for periodic conditions which are not along the direction of the axis, it is possible to find ground states that are at a distance greater that $c(\omega)K$ from the boundary imposed by the boundary conditions.

Notice also that it is possible to chose a sequence of these minimizers so that their oscillations diverge, hence, it is impossible to make them converge to a limit even after translating them.

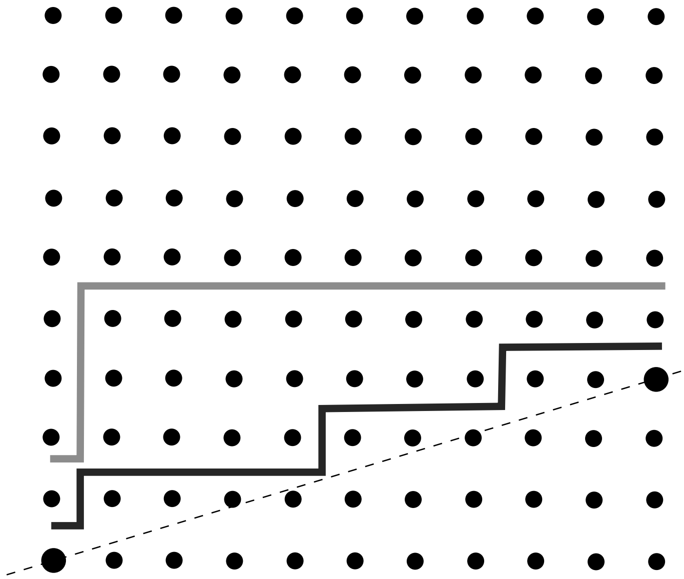

Fig. 3. Non-Birkhoff minimizers and the infimal minimizer for two dimensional classical Ising models.

In contrast, we see that the infimal ground state can be obtained by removing squares with two sides in the interface from the minimizer above (this is a modification that does not change the energy of the interface) as much as possible compatible with the constraint that the interface should lie above the line $\{\omega \cdot x = 0\}$. The interface of these infimal ground states indeed, does not recede more that a fixed constant for the plane and, if we double the period, the minimizer is the same.

Note, however that for some special periodicities – when the plane Π_ω is a coordinate plane, all the minimizers consist only of straight lines. These minimizers are Birkhoff and do not exhibit symmetry breaking. Hence, for the classical Ising model, the symmetry breaking and the Birkhoff property for all periodic minimizers happen or not depending on the orientation of the boundary conditions.

The considerations here should serve as a counterpoint with the analogies with the theory of minimal surfaces mentioned in Remark 2.2. In ref. 6, is is shown all the periodic minimal surfaces are Birkhoff and in ref. 5 it is shown that all periodic minimizers in spin systems and in Dirichlet problems are Birkhoff and that there is no symmetry breaking.

This raises the question of whether there are discrete spin models for which the property that there is no symmetry breaking in ground states and that all ground states are Birkhoff is true. The results of the above papers suggest that this should be true for models which resemble more the continuous models. This suggests that the Birkhoff property for all ground states could be true for models with a longer range interaction (or with several body interactions).

We also note that since the minimizers for a given period are just segments in the horizontal and vertical directions, the average energy can be readily computed and it is

$$\mathcal{A}_{\text{Ising}}(\omega) = |\omega_1| + |\omega_2|$$

This function is, clearly Lipschitz but it is not C^1.

Remark 5.1. A classical problem in statistical mechanics is the study of the interfaces in Ising models for low temperature and the surface tension as a function of the temperature. A collection of classical papers in this area is ref. 18.

Note that even for the Ising model at zero temperature, the fluctuations for interfaces not oriented along the coordinate axis is proportional to the size of the domain,

A related problem is the study of the asymptotic shape of a crystal which has a constraint which is the total number of sites occupied. This goes under the name of the Wulff construction[2,7] and references there. Remarkably similar problems for periodic media have been considered in homogeneization theory. One could expect that, for periodic media, at zero temperature, the shape of a crystal would be a Wulff shape with respect to the average energy $\mathcal{A}$ considered here.

5.2. Layered material

Another example for which it is much easier to create complicated ground states is a layered material in which the layers do not interact.

That is $J_{ij} = -1$ if $|i - j| = 1$ and $e_d \cdot (i - j) = 0$ where e_d is the unit vector along the d coordinate. Otherwise, $J_{i,j} = 0$.

Clearly, a ground state can be obtained by choosing any ground state in each of the layers. Hence, it is possible to chose ground states which are not Birkhoff and which do not converge.

5.3. Antiferromagnetic models, large magnetic fields

We consider the Ising model in the plane, but we include a small periodic patches of antiferromagnetic interactions.

That is, $J_{i,j} = 0$ when $|i-j| \neq 0$. When $|i-j| = 1$, $J_{i,j} = 1$ whenever $d(i/20, \mathbb{Z}^2) < 1/3$, $d(j/10, \mathbb{Z}^2) < 1/3$ and $J_{i,j} = 1$ otherwise.

We see that there cannot be any ground states with plane-like interfaces. Indeed, if we had a configuration as those here, we could create another configuration in which we have flipped a ball around a point in $20\mathbb{Z}^2$. Since the new interface has negative energy, we see that the original configuration cannot be a ground state.

The same effect can be obtained if we concentrate a large and negative magnetic field in balls around the points in $20\mathbb{Z}^2$ and put a magnetic field in the rest so as to adjust the flux condition. Again, we note that one can always lower the energy by creating an interface near any point in $20\mathbb{Z}^2$, so that there cannot be any plane-like ground state.

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